Kernel Quiz

(by default, $x, x' \in \mathcal{X}$, with $\mathcal{X}$ a nonempty set)

$k(x, x') = 1$ $\qquad$ $k(x, x') = -1$ $\qquad$ $k(x, x') = 0$

$k_f(x, x') = f(x)k(x, x')f(x')$, with k p.d. and f a real-valued function

$k_f(x, x') = (-f(x))k(x, x')(-f(x'))$, with k p.d. and f a real-valued function

$k(x, x') = \cos(\angle(x, x'))$, with $\mathcal{X}$ a dot product space

$k(x, x') = \max\{x, x'\}$, for $x, x' \in \mathbb{R}$

$k(x, x') = \min\{x, x'\}$, for $x, x' \in \mathbb{R}_+$

$k(x, x') = k'(f(x), f(x'))$ for $f : \mathcal{X} \to \mathcal{Y}$ and k' p.d. on $\mathcal{Y} \times \mathcal{Y}$

$k(x, x') = \delta_{x,x'}$

$k(x, x') = I_{\{(x,x') | f(x)=f(x')\}}$, for $f : \mathcal{X} \to \mathcal{Y}$ ("equivalence kernel of f")

$k(x, x') = \exp(k'(x, x'))$ for k' p.d.

$k(x, x') = \log(k'(x, x'))$ for k' p.d. $\qquad$ (*can* be p.d. though)

Kernel Quiz, II (by default, $x, x' \in \mathcal{X}$, with $\mathcal{X}$ a nonempty set)

$k(A, B) = P(A \cap B)$ where P is a probability measure

$k(A, B) = P(A \cap B) - P(A)P(B)$ where P is a probability measure

$k(x, x') = \exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right)$ for all σ

$k(x, x') = \exp\left(-\frac{\|x-x'\|^3}{2\sigma^2}\right)$ for all σ

$k(x, x') = \exp\left(-\frac{\|x-x'\|^1}{2\sigma^2}\right)$ for all σ

$k(X, X') = H(X) + H(X') - H(X, X')$ where X, X' discrete RVs (mutual information)

same, with $|\mathcal{X}| \leq 3$ *(Jakobsen, 2013)*

Let $\mathcal{X} = \{1, 2, \dots\}$. Which of the two following kernels is positive definite?
(a) $k(x, x') = lcm(x, x')$ (least common multiple)
(b) $k(x, x') = gcd(x, x')$ (greatest common divisor)

Let $n, m \in \mathbb{N}$.
Let $(p_i)_{i \in \mathbb{N}} = (2, 3, 5, 7, 11, \dots)$ be the sequence of primes.

Let $\phi_i(n)$ be the frequency with which p_i occurs as a prime factor in n.

We have

$$gcd(n, m) = \prod_i \left[p_i^{\min(\phi_i(n), \phi_i(m))} \right],$$

which is p.d. by all the properties we discussed yesterday ($\min(x, y)$, $k(f(x), f(x')$, $exp(k)$, $k \cdot k'$, $\lim_{n \to \infty} k_n(x, x')$,...).

Thomas Provoost found a more fancy proof using the Euler totient function.

This function is also used in (Bhatia, 2006). Bhatia additionally shows that the gcd kernel is infinitely divisible.
http://repository.ias.ac.in/2600/1/375.pdf